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Curvature Tensors and Unified Field Equations on SEX_n

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We study the curvature tensors and field equations in the n-dimensional SE manifold SEX_n. We obtain several basic properties of the vectors S_λ and U_λ and then of the SE curvature tensor and its contractions, such as a generalized Ricci identity, a generalized Bianchi identity, and two variations of the Bianchi identity satisfied by the SE Einstein tensor. Finally, a system of field equations is discussed in SEX_n and one of its particular solutions is constructed and displayed.

1. INTRODUCTION

In Appendix II to his last book Einstein (1950) proposed a new unified field theory that would include both gravitation and electromagnetism. Although the intent of this theory was physical, its exposition was mainly geometrical. It may be characterized as a set of geometrical postulates for the space-time X_4. Although the geometrical consequences of these postulates were not developed very far by Einstein, Hlavatý (1957) gave its mathematical foundation for the first time characterizing Einstein's unified field theory as a set of geometrical postulates for X_4. Since then the geometrical consequences of these postulates have been developed very far by a number of mathematicians and physicists; among them Hlavatý's contributions are the most distinguished.

Generalizing X_4 to n-dimensional generalized Riemannian space X_n, Wrede (1958) studied the Principles A and B given below. But this solution of our (2.7) is not surveyable, probably due to the complexity of the higher dimensions. Mishra (1959) and Chung *et al.* (1981, 1985a,b) also investigated the n-dimensional generalization of principle A, using n-dimensional recurrence relations. However, so far as Principle B is concerned, the solution

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of the system (2.7) has not been obtained in a *surveyable* tensorial form, due to the generality of the higher dimensional space.

Recently Chung *et al.* (to appear) introduced the concept of n-dimensional SE manifold, denoted by SEX_n, imposing the semisymmetric condition (2.24) on X_n, and found the unique representation of the Einstein connection in a beautiful and surveyable form, (2.25). Chung *et al.* (1985a,b) obtained many results concerning SEX_n, such as properties of the submanifold and the hypersubmanifold of SEX_n and generalized fundamental equations on the hypersubmanifold.

The purpose of the present paper is to study the properties of curvature tensors and the field equations in SEX_n and to display a particular solution of the field equations. This paper contains seven sections. Section 2 introduces some preliminary notations, concepts, and results. Section 3 deals with several basic vectors in SEX_n. The next two sections are devoted exclusively to the properties of the SE curvature tensor and its contracted SE curvature tensors. In the last two sections we discuss the field equations in SEX_n and construct a particular solution of them.

All considerations in the present paper are for general $n>1$, unless otherwise stated, and for all possible classes and indices of inertia.

2. PRELIMINARIES

This section is a brief collection of basic concepts, results, and notations needed in subsequent considerations. The detailed proofs are given in Chung *et al.* (1963, 1981, 1985a,b and to appear) and Hlavatý (1957).

2.1. Generalized n-Dimensional Riemannian Manifold

Let X_n be a generalized n-dimensional Riemannian manifold referred to a real coordinate system x^ν, which obeys coordinate transformations $x^\nu \to x^{\nu'}$ for which

$$\mathrm{Det}\left(\frac{\partial x'}{\partial x}\right) \neq 0 \tag{2.1}$$

The space X_n is endowed with a general real, nonsymmetric tensor $g_{\lambda\mu}$, which may be split into a symmetric part $h_{\lambda\mu}$ and a skew-symmetric part $k_{\lambda\mu}$,[3]

$$g_{\lambda\mu} = h_{\lambda\mu} + k_{\lambda\mu} \tag{2.2}$$

where

$$\mathfrak{g} = \mathrm{Det}(g_{\lambda\mu}) \neq 0, \qquad \mathfrak{h} = \mathrm{Det}(h_{\lambda\mu}) \neq 0 \tag{2.3}$$

[3]Throughout the present paper, Greek indices are used for the holonomic components of tensors in X_n. They take the values $1, 2, \cdots, n$ unless stated otherwise and follow the summation convention.

We may define a unique tensor $h^{\lambda\mu}$ by

$$h_{\lambda\mu}h^{\lambda\nu} = \delta_\mu^\nu \tag{2.4}$$

The tensors $h_{\lambda\mu}$ and $h^{\lambda\mu}$ will serve for raising and/or lowering indices of tensors in X_n in the usual manner. In virtue of (2.3) we may also define a unique real tensor $*g^{\lambda\mu}$ by

$$g_{\lambda\mu}\,{}^*g^{\lambda\nu} = g_{\mu\lambda}\,{}^*g^{\nu\lambda} = \delta_\mu^\nu \tag{2.5a}$$

which may be decomposed into

$$^*g^{\lambda\nu} = {}^*h^{\lambda\nu} + {}^*k^{\lambda\nu}; \qquad {}^*h^{\lambda\nu} = {}^*g^{(\lambda\nu)}, \qquad {}^*k^{\lambda\nu} = {}^*g^{(\lambda\nu)} \tag{2.5b}$$

The space X_n is assumed to be connected by a real general connection $\Gamma_{\lambda\mu}^\nu$ with the following transformation rule:

$$\Gamma_{\lambda'\mu'}^{\nu'} = \frac{\partial x^{\nu'}}{\partial x^\alpha}\left(\frac{\partial x^\beta}{\partial x^{\lambda'}}\frac{\partial x^\gamma}{\partial x^{\mu'}}\Gamma_{\beta\gamma}^\alpha + \frac{\partial^2 x^\alpha}{\partial x^{\lambda'}\,\partial x^{\mu'}}\right) \tag{2.6}$$

2.2. Einstein's *n*-Dimensional Unified Field Theory

Einstein's n-dimensional unified field theory is based on the following three principles as indicated by Hlavatý (1957):

Principle A. The algebraic structure is imposed on a generalized n-dimensional Riemannian manifold X_n by a general real tensor $g_{\lambda\mu}$ defined by (2.2).

Principle B. The differential geometric structure is imposed on X_n by the tensor $g_{\lambda\mu}$ by means of the *Einstein connection* $\Gamma_{\lambda\mu}^\nu$ defined by a system of Einstein equations

$$\partial_\omega g_{\lambda\mu} - \Gamma_{\lambda\omega}^\alpha g_{\alpha\mu} - \Gamma_{\omega\mu}^\alpha g_{\lambda\alpha} = 0 \tag{2.7a}$$

or equivalently

$$D_\omega g_{\lambda\mu} = 2S_{\omega\mu}^\alpha g_{\lambda\alpha} \tag{2.7b}$$

where D_ω denotes the symbol of the covariant derivative with respect to $\Gamma_{\lambda\mu}^\nu$ and

$$S_{\omega\mu}^\nu = \Gamma_{(\omega\mu)}^\nu = \tfrac{1}{2}(\Gamma_{\omega\mu}^\nu - \Gamma_{\mu\omega}^\nu) \tag{2.8}$$

Principle C. In order to obtain $g_{\lambda\mu}$ involved in the solution for $\Gamma_{\lambda\mu}^\nu$ in (2.7), certain conditions are imposed, which may be condensed to

$$S_\lambda = S_{\lambda\alpha}^\alpha = 0, \qquad R_{(\mu\lambda)} = \partial_{(\mu}Y_{\lambda)}, \qquad R_{(\mu\lambda)} = 0 \tag{2.9}$$

where Y_λ is an arbitrary vector, and

$$R_{\omega\mu\lambda}^{\nu} = 2(\partial_{(\mu}\Gamma_{|\lambda|\omega)}^\nu + \Gamma_{\alpha(\mu}^\nu\Gamma_{|\lambda|\omega)}^\alpha), \qquad R_{\mu\lambda} = R_{\alpha\mu\lambda}^\alpha \tag{2.10}$$

The following quantities will be used in our further considerations:

$$\mathfrak{t} = \mathrm{Det}(k_{\lambda\mu}) \tag{2.11}$$

$$g = \mathfrak{g}/\mathfrak{h}, \qquad k = \mathfrak{t}/\mathfrak{h} \tag{2.12}$$

$$^{(0)}k_\lambda^\nu = \delta_\lambda^\nu, \quad ^{(p)}k_\lambda^\nu = {}^{(p-1)}k_\lambda^\alpha k_\alpha^\nu \qquad (p = 1, 2, \ldots) \tag{2.13}$$

$$K_0 = 1, \qquad K_p = k_{(\alpha_1}^{\alpha_1} k_{\alpha_2}^{\alpha_2} \cdots k_{\alpha_p)}^{\alpha_p} \qquad (p = 1, 2, \ldots) \tag{2.14}$$

$$\bar{K}_p = K_0 + K_1 + \cdots + K_p \tag{2.15}$$

It is shown in Chung *et al.* (1981) that the following relations hold between the quantities introduced in (2.3) and (2.11)-(2.15):

$$K_n = k \quad \text{if } n \text{ is even:} \qquad K_p = 0 \quad \text{if } p \text{ is odd} \tag{2.16}$$

$$\mathfrak{g} = \mathfrak{h}\bar{K}_n \qquad \text{or} \qquad g = \bar{K}_n \tag{2.17}$$

On the other hand, equations (2.7) can be split into two equations (Hlavatý, 1957)

$$D_\omega h_{\lambda\mu} = 2S_{\omega(\mu}^{\alpha} g_{\lambda)\alpha} \tag{2.18a}$$

$$D_\omega k_{\lambda\mu} = 2S_{\omega(\mu}^{\alpha} g_{\lambda)\alpha} \tag{2.18b}$$

from which we also have

$$D_\omega h^{\lambda\nu} = -2S_{\omega(\alpha}^{\gamma} g_{\beta)\gamma} h^{\beta\nu} h^{\alpha\lambda} \tag{2.18c}$$

A procedure similar to Christoffel's elimination applied to (2.18a) yields that if equations (2.7) admit a solution $\Gamma_{\lambda\mu}^\nu$, it must be of the form (Hlavatý, 1957)

$$\Gamma_{\lambda\mu}^\nu = \{{}_{\lambda\mu}^{\nu}\} + S_{\lambda\mu}^\nu + U_{\lambda\mu}^\nu \tag{2.19}$$

where

$$U_{\lambda\mu}^\nu = 2h^{\nu\alpha} S_{\alpha(\lambda}{}^{\beta} k_{\mu)\beta} \tag{2.20}$$

and $\{{}_{\lambda\mu}^{\nu}\}$ are Christoffel symbols defined by $h_{\lambda\mu}$.

Using the concepts of basic vectors and scalars, Chung *et al.* (1985a,b) derived the following recurrence relation:

$$\sum_{f=0}^{n-\sigma} K_f {}^{(n+p-f)}k_\lambda^\nu = 0 \qquad (p = 0, 1, 2, \cdots) \tag{2.21}$$

$$\sigma = \begin{cases} 0, & \text{if } \ n \text{ is even} \\ 1, & \text{if } \ n \text{ is odd} \end{cases} \tag{2.22}$$

Using (2.21), it has also been shown (Chung *et al.*, 1981, 1985a,b) that the n-dimensional representations of the tensors $^*h^{\lambda\nu}$, $^*k^{\lambda\nu}$, $^*h_{\lambda\mu}$, and $^*k_{\lambda\mu}$ are given by

$$^*h^{\lambda\nu} = \frac{1}{g}\sum_{p=0}^{n-1}(K_0\,{}^{(p)}k^{\lambda\nu} + K_2\,{}^{(p-2)}k^{\lambda\nu} + \cdots + K_{p-2}\,{}^{(2)}k^{\lambda\nu} + K_p h^{\lambda\nu}) \tag{2.23a}$$

$$^*k^{\lambda\nu} = \frac{1}{g}\sum_{p=2}^{n}(K_0\,{}^{(p-1)}k^{\lambda\nu} + K_2\,{}^{(p-3)}k^{\lambda\nu} + \cdots + K_{p-4}\,{}^{(3)}k^{\lambda\nu} + K_{p-2}\,k^{\lambda\nu}) \tag{2.23b}$$

$$^*h_{\lambda\mu} = h_{\lambda\mu} - {}^{(2)}k_{\lambda\mu}, \qquad {}^*k_{\lambda\mu} = k_{\lambda\mu} - {}^{(3)}k_{\lambda\mu} \tag{2.23c}$$

2.3. *n*-Dimensional *SE* Manifold SEX$_n$

A connection $\Gamma^{\nu}_{\lambda\mu}$ is said to be *semisymmetric* if its torsion tensor $S^{\nu}_{\lambda\mu}$ is of the form

$$S^{\nu}_{\lambda\mu} = 2\delta^{\nu}_{(\lambda}X_{\mu)} \tag{2.24}$$

for an arbitrary vector X_μ. A connection that is both semisymmetric and Einstein is called an *SE connection*, and a generalized n-dimensional Riemannian manifold X_n on which the differential geometric structure is imposed by $g_{\lambda\mu}$ through a SE connection is called an *n-dimensional SE manifold, denoted by SEX$_n$*.

It has been shown (Chung *et al.*, to appear) that there always exists a unique n-dimensional SE connection $\Gamma^{\nu}_{\lambda\mu}$ of the form

$$\Gamma^{\nu}_{\lambda\mu} = \{^{\nu}_{\lambda\mu}\} + 2k^{\nu}_{(\lambda}X_{\mu)} + 2\delta^{\nu}_{(\lambda}X_{\mu)} \tag{2.25}$$

for a unique vector X_λ given by

$$X_\lambda = \frac{1}{n-1}\,{}^*h^{\alpha\beta}\,\nabla_\alpha k_{\beta\lambda} \tag{2.26}$$

where ∇_λ is the symbolic vector of the covariant derivative with respect to $\{^{\nu}_{\lambda\mu}\}$.

3. THE VECTORS X_λ, S_λ, AND U_λ

Agreement 3.1. Our further considerations in the present paper are exclusively restricted to the n-dimensional SE manifold SEX$_n$, $n > 1$.

In this section we investigate several properties of the vector X_λ given by (2.26) and the vectors

$$S_\lambda = S^{\alpha}_{\lambda\alpha}, \qquad U_\lambda = U^{\alpha}_{\lambda\alpha} \tag{3.1}$$

The following theorems are needed in our further considerations.

Theorem 3.2. In SEX_n the vectors S_λ and U_λ are given by

$$S_\lambda = (1-n)X_\lambda \tag{3.2}$$

$$U_\lambda = k_\lambda^\alpha X_\alpha = \tfrac{1}{2}\partial_\lambda \ln g \tag{3.3}$$

Proof. Putting $\mu = \nu$ in (2.24), we have (3.2). Similarly, the first relation of (3.3) may be obtained by putting $\mu = \nu$ in

$$U_{\lambda\mu}^\nu = 2k_{(\lambda}^\nu X_{\mu)} \tag{3.4}$$

which is a result of (2.20) and (2.24). In order to prove the second relation of (3.3), multiply by $^*g^{\lambda\mu}$ on sides of (2.7a) and make use of (2.5a) to get

$$\partial_\omega \ln \mathfrak{g} - \Gamma_{\alpha\omega}^\alpha - \Gamma_{\omega\alpha}^\alpha = 0 \tag{3.5a}$$

or

$$\partial_\omega \ln \mathfrak{g} + 2S_\omega - 2\Gamma_{\omega\alpha}^\alpha = 0 \tag{3.5b}$$

On the other hand, in virtue of the classical result

$$\left\{ {}_{\alpha\mu}^{\alpha} \right\} = \tfrac{1}{2}\partial_\mu \ln \mathfrak{h} \tag{3.6}$$

we have

$$\Gamma_{\omega\alpha}^\alpha = \tfrac{1}{2}\partial_\omega \ln \mathfrak{h} + S_\omega + U_\omega \tag{3.5c}$$

The second relation of (3.3) immediately follows from (3.5b) and (3.5c). ■

Remark 3.3. In virtue of (3.3) and (3.4), we note that

$$X_\lambda = 0 \quad \text{implies} \quad S_\lambda = 0 \quad \text{and} \quad U_\lambda = 0$$

Now, introduce the following abbreviation for an arbitrary real vector Y_λ:

$$\overset{(p)}{Y}_\lambda = {}^{(p)}k_\lambda^\alpha Y_\alpha \qquad (p = 0, 1, 2, \cdots) \tag{3.7a}$$

Then, since

$${}^{(p)}k_{\lambda\mu} = (-1)^p\, {}^{(p)}k_{\mu\lambda},$$

we have

$$\overset{(p)}{Y}{}^\nu = (-1)^p\, {}^{(p)}k_\alpha^\nu Y^\alpha \qquad (p = 0, 1, 2, \cdots) \tag{3.7b}$$

and in particular

$$\overset{(0)}{Y}_l = Y_\lambda, \qquad \overset{(0)}{Y}{}^\nu = Y^\nu \tag{3.7c}$$

Employing this abbreviation, we have the following sequence of theorems, which will be needed in our subsequent considerations:

Theorem 3.4. In SEX$_n$ the following relations hold:

$$\overset{(p)}{X}_\lambda = \overset{(p-1)}{U}_\lambda, \qquad \overset{(p)}{S}_\lambda = (1-n)\overset{(p-1)}{U}_\lambda \quad (p = 1, 2, \cdots) \tag{3.8a}$$

$$\overset{(p)}{X}{}^\nu = -\overset{(p-1)}{U}{}^\nu, \qquad \overset{(p)}{S}{}^\nu = (n-1)\overset{(p-1)}{U}{}^\nu \tag{3.8b}$$

Proof. In virtue of (2.13), (3.2), (3.7), and the skew symmetry of $k_{\lambda\mu}$, our assertions in this theorem are immediate consequences of (3.3). ■

Theorem 3.5. In SEX$_n$ the following relations hold:

$$\overset{(p)}{U}_\alpha \overset{(q)}{X}{}^\alpha = \overset{(p)}{U}_\alpha \overset{(q)}{S}{}^\alpha = 0 \qquad \text{if } p+q+1 \text{ is odd } (p, q = 0, 1, 2, \cdots) \tag{3.9a}$$

In particular, we have

$$\overset{(p)}{U}_\alpha X^\alpha = \overset{(p)}{U}_\alpha S^\alpha = 0 \qquad \text{if } p = 0, 2, 4, \cdots \tag{3.9b}$$

Proof. The relations (3.7a), (3.7b), and (3.8a) give

$$\overset{(p)}{U}_\alpha \overset{(q)}{X}{}^\alpha = \overset{(p+1)}{X}_\alpha \overset{(q)}{X}{}^\alpha = (-1)^q \, {}^{(p+q+1)}k_{\alpha\beta} X^\alpha X^\beta \tag{3.10}$$

Our assertion (3.9a) follows from (3.10) in virtue of the skew symmetry of $k_{\lambda\mu}$ and (3.2). The relation (3.9b) is an immediate consequence of (3.9a). ■

Theorem 3.6. In SEX$_n$ the following relations hold:

$$U^\alpha_{\lambda\mu} X_\alpha = 2U_{(\lambda} X_{\mu)} = 2\overset{(1)}{X}_{(\lambda} X_{\mu)} \tag{3.11a}$$

$$U^\alpha_{\lambda\mu} S_\alpha = 2(1-n) U_{(\lambda} X_{\mu)} = 2(1-n)\overset{(1)}{X}_{(\lambda} X_{\mu)} \tag{3.11b}$$

$$U^\alpha_{\lambda\mu} U_\alpha = 2\overset{(1)}{U}_{(\lambda} X_{\mu)} = 2\overset{(2)}{X}_{(\lambda} X_{\mu)} \tag{3.11c}$$

Proof. In virtue of (3.4) and (3.7a), we have

$$U^\alpha_{\lambda\mu} \overset{(p)}{Y}_\alpha = 2k^\alpha_{(\lambda} X_{\mu)} \overset{(p)}{Y}_\alpha = 2\overset{(p+1)}{Y}_{(\lambda} X_{\mu)} \qquad (p = 0, 1, 2, \cdots) \tag{3.12}$$

Our assertions (3.11) can be easily shown from (3.12). ■

Theorem 3.7. In SEX_n the vector $\overset{(p)}{X}_\lambda$ satisfies the following recurrence relation:

$$\sum_{f=0}^{n-\sigma} K_f \overset{(n+p-f)}{X_\lambda} = 0 \qquad (p=0,1,2,\cdots) \tag{3.13}$$

Proof. Multiplying by X_ν on both sides of (2.21) and making use of (3.7a), we have (3.13). ∎

Theorem 3.8. In SEX_n the following relations hold:

$$D_\lambda X_\mu = \nabla_\lambda X_\mu - 2U_{(\lambda}X_{\mu)} \tag{3.14a}$$

$$D_{(\lambda}X_{\mu)} = \nabla_{(\lambda}X_{\mu)} = \partial_{(\lambda}X_{\mu)} \tag{3.14b}$$

$$\nabla_{(\lambda}U_{\mu)} = 0, \qquad D_{(\lambda}U_{\mu)} = 2U_{(\lambda}X_{\mu)} = 2\overset{(1)}{X}_{(\lambda}X_{\mu)} \tag{3.14c}$$

Proof. In virtue of (2.25) and (3.12), we have

$$D_{(\lambda}Y_{\mu)} = \partial_{(\lambda}Y_{\mu)} + 2Y_{(\lambda}X_{\mu)} \tag{3.15}$$

for an arbitrary vector Y_λ. Our assertions follows easily from (3.3) and (3.15). ∎

4. THE n-DIMENSIONAL SE CURVATURE TENSOR

This section is devoted to the study of the n-dimensional SE curvature tensor $R_{\omega\mu\lambda}{}^{\nu}$ defined by the SE connection $\Gamma_{\lambda\mu}^{\nu}$ and of some identities involving the tensor $R_{\omega|\lambda}^{\nu}$.

Having found the SE connection in the form (2.25), as shown in the following theorem, we may derive the representation of the SE curvature tensor $R_{\omega|\lambda}^{\nu}$ as a function of X_λ, $g_{\lambda\mu}$, and their first two derivatives by simply substituting (2.25) into (2.10).

Theorem 4.1. The n-dimensional SE curvature tensor $R_{\omega\mu\lambda}{}^{\nu}$ in SEX_n is given by

$$R_{\omega\mu\lambda}{}^{\nu} = H_{\omega\mu\lambda}{}^{\nu} + \underset{1}{R}{}_{\omega\mu\lambda}{}^{\nu} + \underset{2}{R}{}_{\omega\mu\lambda}{}^{\nu} \tag{4.1}$$

where

$$H_{\omega\mu\lambda}{}^{\nu} = 2\partial_{(\mu}\{{}_{|\lambda|}{}^{\nu}{}_{\omega)}\} + 2\{{}_{\alpha(\mu}{}^{\nu}\}\{{}_{|\lambda|}{}^{\alpha}{}_{\omega)}\} \tag{4.2a}$$

$$\underset{1}{R}{}_{\omega\mu\lambda}{}^{\nu} = 2\delta_\lambda^\nu \partial_{(\mu}X_{\omega)} + 2\delta_{(\mu}^\nu \nabla_{\omega)}X_\lambda + 2\nabla_{(\mu}U_{\omega)\lambda}^{\nu} \tag{4.2b}$$

$$\underset{2}{R}{}_{\omega\mu\lambda}{}^{\nu} = 2\,{}^*h_{(\mu}^{\nu}X_{\omega)}X_\lambda + 2g_{(\omega}^{\nu}U_{\mu)}X_\lambda + 2g_{(\omega}^{\nu}X_{\mu)}U_\lambda \tag{4.2c}$$

Proof. Substitute (2.25) into (2.10) and make use of (4.2a) to obtain

$$
\begin{aligned}
R^{\nu}_{\omega\mu\lambda} &= 2\partial_{(\mu}(\{{}^{\nu}_{\omega)\lambda}\} + X_{\omega)}\delta^{\nu}_{\lambda} - \delta^{\nu}_{\omega)}X_{\lambda} + U^{\nu}_{\omega)\lambda}) \\
&\quad + 2(\{{}^{\nu}_{\alpha(\mu}\} + \delta^{\nu}_{\alpha}X_{(\mu} - X_{\alpha}\delta^{\nu}_{(\mu} + U^{\nu}_{\alpha(\mu})(\{{}^{\alpha}_{\omega)\lambda}\} + X_{\omega)}\delta^{\alpha}_{\lambda} - \delta^{\alpha}_{\omega)}X_{\lambda} + U^{\alpha}_{\omega)\lambda}) \\
&= H^{\nu}_{\omega\mu\lambda} + 2\delta^{\nu}_{\lambda}\partial_{(\mu}X_{\omega)} + 2(\delta^{\nu}_{(\mu}\partial_{\omega)}X_{\lambda} - \delta^{\nu}_{(\mu}\{{}^{\alpha}_{\omega)\lambda}\}X_{\alpha}) \\
&\quad + 2(\partial_{(\mu}U^{\nu}_{\omega)\lambda} + \{{}^{\alpha}_{\lambda(\omega}\}U^{\nu}_{\mu)\alpha} + \{{}^{\nu}_{\alpha(\mu}\}U^{\alpha}_{\omega)\lambda}) \\
&\quad + 2(\delta^{\nu}_{(\mu}X_{\omega)}X_{\lambda} - X_{\alpha}\delta^{\nu}_{(\mu}U^{\alpha}_{\omega)\lambda} + U^{\nu}_{\alpha(\mu}U^{\alpha}_{\omega)\lambda})
\end{aligned}
\tag{4.3}
$$

Clearly the sum of the second, third, and fourth terms on the right-hand side of (4.3) is $\underset{1}{R}{}^{\nu}_{\omega\mu\lambda}$. On the other hand, we have

$$g^{\nu}_{\mu} = \delta^{\nu}_{\mu} - k^{\nu}_{\mu} \tag{4.4a}$$

$$*h^{\nu}_{\mu} = \delta^{\nu}_{\mu} - {}^{(2)}k^{\nu}_{\mu} \tag{4.4b}$$

where use has been made of (2.2) and (2.23c). Now, substituting (3.4) into the fifth term on the right-hand side of (4.3) and making use of (4.4), we note that the fifth term is equal to $\underset{2}{R}{}^{\nu}_{\omega\mu\lambda}$ in virtue of (4.2c). Hence our proof is completed. ■

Theorem 4.2. The SE curvature tensor $R^{\nu}_{\omega\mu\lambda}$ obeys the following identities:

$$R^{\nu}_{\omega\mu\lambda} = R^{\nu}_{(\omega\mu)\lambda} \tag{4.5}$$

$$R^{\nu}_{(\omega\mu\lambda)} = 4\delta^{\nu}_{(\lambda}\partial_{\mu}X_{\omega)} \tag{4.6}$$

Proof. Equation (4.5) follows immediately from (2.10). In order to prove (4.6), we use (4.1) to obtain

$$R^{\nu}_{(\omega\mu\lambda)} = H^{\nu}_{(\omega\mu\lambda)} + \underset{1}{R}{}^{\nu}_{(\omega\mu\lambda)} + \underset{2}{R}{}^{\nu}_{(\omega\mu\lambda)} \tag{4.7}$$

In virtue of (4.2) we have

$$H^{\nu}_{(\omega\mu\lambda)} = \underset{2}{R}{}^{\nu}_{(\omega\mu\lambda)} = 0, \qquad \underset{1}{R}{}^{\nu}_{(\omega\mu\lambda)} = 4\delta^{\nu}_{(\mu}\partial_{\omega}X_{\lambda)} \tag{4.8}$$

The identity (4.6) follows by substitution of (4.8) into (4.7).

The following two theorems are immediate consequences of Hlavatý's results (Hlavatý, 1957, p. 129):

$$
\begin{aligned}
2D_{(\omega}D_{\mu)}T^{\nu_1\cdots\nu_p}_{\lambda_1\cdots\lambda_q} = &-\sum_{\alpha=1}^{p} T^{\nu_1\cdots\nu_{\alpha-1}\xi\nu_{\alpha+1}\cdots\nu_p}_{\lambda_1\cdots\lambda_q} R^{\nu_\alpha}_{\omega\mu\xi} \\
&+ \sum_{\beta=1}^{q} T^{\nu_1\cdots\nu_p}_{\lambda_1\cdots\lambda_{\beta-1}\xi\lambda_{\beta+1}\cdots\lambda_q} R^{\xi}_{\omega\mu\lambda_\beta} \\
&+ 2S^{\alpha}_{\omega\mu}D_{\alpha}T^{\nu_1\cdots\nu_p}_{\lambda_1\cdots\lambda_q}
\end{aligned}
\tag{4.9}
$$

$$D_{(\xi}R^{\nu}_{\omega\mu)\lambda} = -2S^{\beta}_{(\xi\omega}R^{\nu}_{\mu)\beta\lambda} \tag{4.10}$$

which hold on a manifold to which an Einstein connection is connected. ■

Theorem 4.3. (Generalized Ricci identity in SEX_n.) The SE curvature tensor $R^{\nu}_{\omega\mu\lambda}$ in SEX_n satisfies the following identity:

$$2D_{(\omega}D_{\mu)}T^{\nu_1\cdots\nu_p}_{\lambda_1\cdots\lambda_q} = -\sum_{\alpha=1}^{p} T^{\nu_1\cdots\nu_{\alpha-1}\xi\nu_{\alpha+1}\cdots\nu_p}_{\lambda_1\cdots\lambda_q} R^{\nu_\alpha}_{\omega\mu\xi}$$

$$+\sum_{\beta=1}^{q} T^{\nu_1\cdots\nu_p}_{\lambda_1\cdots\lambda_{\beta-1}\xi\lambda_{\beta+1}\cdots\lambda_q} R^{\xi}_{\omega\mu\lambda_\beta}$$

$$-4X_{(\omega}D_{\mu)}T^{\nu_1\cdots\nu_p}_{\lambda_1\cdots\lambda_q} \tag{4.11}$$

Proof. Making use of (2.24), we see that (4.11) is a direct consequence of (4.9). ■

Theorem 4.4. (Generalized Bianchi identity in SEX_n.) The SE curvature tensor $R^{\nu}_{\omega\mu\lambda}$ in SEX_n satisfies the following identity:

$$D_{(\xi}R^{\nu}_{\omega\mu)\lambda} = -4X_{(\xi}H^{\nu}_{\omega\mu)\lambda} + M^{\nu}_{(\xi\omega\mu)\lambda} \tag{4.12}$$

where

$$\tfrac{1}{8}M^{\nu}_{\xi\omega\mu\lambda} = (\delta^{\nu}_{\lambda}X_\xi\partial_\omega X_\mu + X_\xi\delta^{\nu}_{\omega}\nabla_\mu X_\lambda + X_\xi\nabla_\omega U^{\nu}_{\mu l}) + g^{\nu}_{\xi}X_\omega U_\mu X_\lambda \tag{4.13}$$

Proof. In virtue of (2.24) and (4.1), the identity (4.10) may be rewritten as

$$D_{(\xi}R^{\nu}_{\omega\mu)\lambda} = -2S^{\beta}_{(\xi\omega}H^{\nu}_{\mu)\beta\lambda} - 2S^{\beta}_{(\xi\omega}\underset{1}{R}^{\nu}_{\mu)\beta\lambda} - 2S^{\beta}_{(\xi\omega}\underset{2}{R}^{\nu}_{\mu)\beta\lambda}$$

$$= -4X_{(\omega}H^{\nu}_{\mu\xi)\lambda} - 4X_{(\xi}\underset{1}{R}^{\nu}_{\omega\mu)\lambda} - 4X_{(\xi}\underset{2}{R}^{\nu}_{\omega\mu)\lambda} \tag{4.14}$$

In virtue of (4.2b) the second term on the right-hand side of (4.14) may be expressed in the form

$$X_{(\xi}\underset{1}{R}^{\nu}_{\omega\mu)\lambda} = 2(\delta^{\nu}_{\lambda}X_{(\xi}\partial_\mu X_{\omega)} + X_{(\xi}\delta^{\nu}_{\mu}\nabla_{\omega)}X_\lambda + X_{(\xi}\nabla_\mu U^{\nu}_{\omega)\lambda}) \tag{4.15a}$$

The relation (4.2c), together with (4.4a), enables us to write the third term on the right-hand side of (4.14) as follows:

$$X_{(\xi}\underset{2}{R}^{\nu}_{\omega\mu)\lambda} = -2(\delta^{\nu}_{(\xi}X_\omega U_{\mu)} - k^{\nu}_{(\xi}X_\omega U_{\mu)})X_\lambda$$

$$= -2g^{\nu}_{(\xi}X_\omega U_{\mu)}X_\lambda \tag{4.15b}$$

We now substitute (4.15a) and (4.15b) into (4.14) and make use of (4.13) to complete the proof of (4.12). ■

5. THE CONTRACTED SE CURVATURE TENSORS

This section is devoted to the study of the contracted n-dimensional SE curvature tensors of the SE connection $\Gamma^{\nu}_{\lambda\mu}$ and of some identities involving them.

The tensors

$$R_{\mu\lambda} = R^{\alpha}_{\alpha\mu\lambda}, \qquad V_{\omega\mu} = R^{\alpha}_{\omega\mu\alpha} \tag{5.1}$$

are called *the first and second contracted SE curvature tensors* of the SE connection $\Gamma^{\nu}_{\lambda\mu}$, respectively. They also appear as functions of $g_{\lambda\mu}$ and its first two derivatives.

Theorem 5.1. The second contracted SE curvature tensor $V_{\omega\mu}$ in SEX$_n$ is a curl of the vector S_λ. That is,

$$V_{\omega\mu} = 2\partial_{(\omega}S_{\mu)} \tag{5.2}$$

Proof. Putting $\lambda = \nu = \alpha$ in (4.1), we have

$$V_{\omega\mu} = H^{\alpha}_{\omega\mu\alpha} + \underset{1}{R}^{\alpha}_{\omega\mu\alpha} + \underset{2}{R}^{\alpha}_{\omega\mu\alpha} \tag{5.3}$$

In virtue of (3.2), (3.14b), and (4.4), the relations (4.2) give

$$H^{\alpha}_{\omega\mu\alpha} = \underset{2}{R}^{\alpha}_{\omega\mu\alpha} = 0$$

$$\underset{1}{R}^{\alpha}_{\omega\mu\alpha} = 2n\partial_{(\mu}X_{\omega)} + 2\nabla_{(\omega}X_{\mu)} = 2(1-n)\partial_{(\omega}X_{\mu)} = 2\partial_{(\omega}S_{\mu)}$$

which together with (5.3) proves (5.2). ■

Theorem 5.2. The first contracted SE curvature tensor $R_{\mu\lambda}$ in SEX$_n$ is given by

$$\begin{aligned} R_{\mu\lambda} = {} & H_{\mu\lambda} + 2\partial_{(\mu}X_{\lambda)} + \nabla_{\mu}T_{\lambda} - \nabla_{\alpha}U^{\alpha}_{\mu\lambda} \\ & + (1-n-2K_2)X_{\mu}X_{\lambda} + U_{\mu}U_{\lambda} - 2U_{(\mu}S_{\lambda)} \end{aligned} \tag{5.4}$$

where

$$H_{\mu\lambda} = H^{\alpha}_{\alpha\mu\lambda} \tag{5.5a}$$

$$T^{\nu}_{\lambda\mu} = S^{\nu}_{\lambda\mu} + U^{\nu}_{\lambda\mu}, \qquad T_{\lambda} = T^{\alpha}_{\lambda\alpha} = S_{\lambda} + U_{\lambda} \tag{5.5b}$$

Proof. Putting $\omega = \nu = \alpha$ in (4.1) and making use of (5.5a), we have

$$R_{\mu\lambda} = H_{\mu\lambda} + \underset{1}{R}^{\alpha}_{\alpha\mu\lambda} + \underset{2}{R}^{\alpha}_{\alpha\mu\lambda} \tag{5.6}$$

Now, the relation (4.2b), together with (3.2), (3.14b), and (5.5b), gives

$$\underset{1}{R}{}^{\alpha}_{\alpha\mu\lambda}=2\partial_{(\mu}X_{\lambda)}+(1-n)\nabla_{\mu}X_{\lambda}+\nabla_{\mu}U_{\lambda}-\nabla_{\alpha}U^{\alpha}_{\mu\lambda}$$

$$=2\partial_{(\mu}X_{\lambda)}+\nabla_{\mu}T_{\lambda}-\nabla_{\alpha}U^{\alpha}_{\mu\lambda} \tag{5.7a}$$

On the other hand, in virtue of (2.14), (3.2), (3.7a), (3.8a), and (4.4), it follows from (4.2c) that

$$\underset{2}{R}{}^{\alpha}_{\alpha\mu\lambda}=2(\delta^{\alpha}_{(\mu}-{}^{(2)}k^{\alpha}_{(\mu})X_{\alpha)}X_{\lambda}+2(\delta^{\alpha}_{(\alpha}-k^{\alpha}_{(\alpha})U_{\mu)}X_{\lambda}+2(\delta^{\alpha}_{(\alpha}-k^{\alpha}_{(\alpha})X_{\mu)}U_{\lambda}$$

$$=(1-n-2K_2)X_{\mu}X_{\lambda}-2U_{(\mu}S_{\lambda)}+U_{\mu}U_{\lambda} \tag{5.7b}$$

Our assertion follows immediately from (5.6), (5.7a) and (5.7b). ■

Remark 5.3. In virtue of (2.26), (3.2), (3.3), and (3.4), we note from (5.4) that $R_{\mu\lambda}$ is a function of $g_{\lambda\mu}$ and its first two derivatives.

Theorem 5.4. If X_{λ} is not a gradient vector, the tensor $R_{\mu\lambda}$ is symmetric only when $n=3$.

Proof. The expression (5.4) may be rewritten as

$$R_{\mu\lambda}=H_{\mu\lambda}-2\nabla_{(\mu}X_{\lambda)}+(3-n)\nabla_{\mu}X_{\lambda}+\nabla_{\mu}U_{\lambda}-\nabla_{\alpha}U^{\alpha}_{\mu\lambda}$$
$$+(1-n-2K_2)X_{\mu}X_{\lambda}+U_{\mu}U_{\lambda}-2U_{(\mu}S_{\lambda)} \tag{5.8}$$

where use has been made of (3.14b) and (5.5b). Since $\nabla_{\mu}U_{\lambda}=\nabla_{\lambda}U_{\mu}$ in virtue of (3.14b), we have

$$R_{(\mu\lambda)}=0 \rightleftarrows (3-n)\nabla_{(\mu}X_{\lambda)}=(3-n)\partial_{(\mu}X_{\lambda)}=0$$

from which our assertion follows. ■

Remark 5.5. In Theorem 5.4 we excluded the case $\partial_{(\mu}X_{\lambda)}=0$, because the assumption that *the vector X_{λ} is not a gradient vector* is necessary in the discussion of the field equations in X_n.

Theorem 5.6. The contracted SE curvature tensors in SEX_n are related by

$$2R_{(\mu\lambda)}=4\partial_{(\mu}X_{\lambda)}+V_{\mu\lambda} \tag{5.9}$$

Proof. In virtue of (3.2), (3.14b), (5.2), and (5.8), we can prove relation (5.9) in the following way:

$$2R_{(\mu\lambda)}=2(3-n)\partial_{(\mu}X_{\lambda)}=2(1-n)\partial_{(\mu}X_{\lambda)}+4\partial_{(\mu}X_{\lambda)}$$
$$=2\partial_{(\mu}S_{\lambda)}+4\partial_{(\mu}X_{\lambda)}=V_{\mu\lambda}+4\partial_{(\mu}X_{\lambda)} \quad ■$$

Remark 5.7. An alternative proof of Theorem 5.6 may be obtained by putting $\lambda = \nu = \alpha$ in (4.6) to derive

$$V_{\omega\mu} - R_{\mu\omega} + R_{\omega\mu} = 4\partial_{(\omega}S_{\mu)} + 4\partial_{(\omega}X_{\mu)}$$

Our next task is to obtain a generalization of the classical identity

$$\nabla_\alpha E^\alpha_\mu = 0 \tag{5.10}$$

where[4]

$$H = h^{\alpha\beta}H_{\alpha\beta}, \qquad E^\nu_\mu = H^\nu_\mu - \tfrac{1}{2}\delta^\nu_\mu H \tag{5.11}$$

The quantities

$$R = h^{\alpha\beta}R_{\alpha\beta}, \qquad G^\nu_\mu = R^\nu_\mu - \tfrac{1}{2}\delta^\nu_\mu R \tag{5.12}$$

will be referred to as the *SE curvature invariant* and *SE Einstein tensor of SEX$_n$*, respectively. First we need the following two theorems.

Theorem 5.8. In SEX$_n$ we have

$$D_\omega h^{\lambda\mu} = -2X^{(\lambda}g^{\nu)}_\omega + 2X_\omega h^{\lambda\nu} \tag{5.13a}$$

$$D_\alpha h^{\lambda\alpha} = S^\lambda + U^\lambda \tag{5.13b}$$

Proof. In virtue of (2.18c), (2.24), and (4.1a), we can prove the relation (5.13a) in the following way:

$$\begin{aligned} D_\omega h^{\lambda\mu} &= -2(\delta^\gamma_\omega X_{(\alpha} - X_\omega \delta^\gamma_{(\alpha})g_{\beta)\gamma}h^{\beta\nu}h^{\alpha\lambda} \\ &= 2(-X_{(\alpha}g_{\beta)\omega} + X_\omega h_{\alpha\beta})h^{\beta\nu}h^{\alpha\lambda} \\ &= -2X^{(\lambda}(\delta^{\nu)}_\omega - k^{\nu)}_\omega) + 2X_\omega h^{\lambda\nu} \\ &= -2X^{(\lambda}g^\nu_\omega + 2X_\omega h^{\lambda\nu} \end{aligned}$$

The relation (5.13b) is a direct consequence of (5.13a). ■

Theorem 5.9. In SEX$_n$ we have

$$R = H + \nabla_\alpha T^\alpha - \nabla_\alpha U^{\alpha\beta}_\beta + (1 - n - 2K_2)X + U \tag{5.14a}$$

$$D_\alpha R^\alpha_\mu = \nabla_\alpha R^\alpha_\mu + (U_\alpha - nX_\alpha)R^\alpha_\mu + RX_\mu - U^\beta_{\mu\alpha}R^\alpha_\beta \tag{5.14b}$$

where

$$X = X_\alpha X^\alpha, \qquad U = U_\alpha U^\alpha \tag{5.14c}$$

[4]The tensor E^ν_μ is called the Einstein tensor. This tensor is of fundamental importance because its divergence vanishes identically in virtue of (5.10).

Proof. The representation (5.14a) follows from (5.4) in virtue of (3.9), (3.14b), (5.11), and (5.14c). The relation (5.14b) may be shown in the following way in virtue of (2.24), (3.2), and (5.5b):

$$\begin{aligned} D_\alpha R^\alpha_\mu &= \partial_\alpha R^\alpha_\mu + \Gamma^\alpha_{\beta\alpha} R^\beta_\mu - \Gamma^\beta_{\mu\alpha} R^\alpha_\beta \\ &= \nabla_\alpha R^\alpha_\mu + T_\beta R^\beta_\mu - S^\beta_{\mu\alpha} R^\alpha_\beta - U^\beta_{\mu\alpha} R^\alpha_\beta \\ &= \nabla_\alpha R^\alpha_\mu + (T_\alpha - X_\alpha) R^\alpha_\mu + R X_\mu - U^\beta_{\mu\alpha} R^\alpha_\beta \\ &= \nabla_\alpha R^\alpha_\mu + (U_\alpha - n X_\alpha) R^\alpha_\mu + R X_\mu - U^\beta_{\mu\alpha} R^\alpha_\beta \end{aligned}$$

Now we are ready to prove the following generalization of (5.10). ■

Theorem 5.10a. (A variation of the generalized Bianchi identity in SEX_n.) The SE Einstein tensor G^ν_μ satisfies the following identity in SEX_n:

$$D_\alpha G^\alpha_\mu = P_\mu - \tfrac{1}{2}\partial_\mu M \tag{5.15a}$$

where

$$P_\mu = \nabla_\alpha (R^\alpha_\mu - H^\alpha_\mu) + (U_\alpha - n X_\alpha) R^\alpha_\mu + R X_\mu - U^\beta_{\mu\alpha} R^\alpha_\beta \tag{5.15b}$$

$$M = \nabla_\alpha T^\alpha - \nabla_\alpha U^{\alpha\beta}_\beta + (1 - n - 2K_2) X + U \tag{5.15c}$$

Proof. The proof of (5.15a) follows easily from (5.12) in virtue of (5.10), (5.14), (5.15b) and (5.15c). ■

Remark 5.11. Several earlier authors (e.g., Bose 1953; Einstein, 1955; Lichnerowicz, 1955; Schrödinger, 1949; Winogradski, 1956) tried to generalize (5.10) on a manifold to which an Einstein connection is connected, but their results are cumbersome. Note that our result (5.15) in the above theorem, which holds in SEX_n, is a very handy and surveyable tensorial form.

Theorem 5.10b. (A variation of the generalized Bianchi identity in SEX_n.) The SE Einstein tensor G^ν_μ satisfies the following identity in SEX_n:

$$2D_\alpha G^\alpha_\mu = X^\alpha Q_{\mu\alpha} - U^\alpha R_{\mu\alpha} - 2RX_\mu - 3h^{\omega\lambda} M^\beta_{(\beta\omega\mu)\lambda} + h^{\omega\lambda} D_\beta (R_{\omega\mu(\lambda\alpha)} h^{\alpha\beta}) \tag{5.16a}$$

where

$$Q_{\mu\alpha} = (3-n) R_{\mu\alpha} + R_{\alpha\mu} - 8E_{\mu\alpha} - R^\beta_{\alpha\mu\gamma} k^\gamma_\beta - R_{\alpha\beta} k^\beta_\mu \tag{5.16b}$$

Proof. The proof of this assertion is based on the generalized Bianchi identity (4.12). It may be written in the form

$$
\begin{aligned}
&-D_\xi(R_{\omega\mu\alpha\lambda}h^{\nu\alpha})+D_\omega R^{\nu}_{\mu\xi\lambda}+D_\mu R^{\nu}_{\xi\omega\lambda}\\
&\quad=-12X_{(\xi}H^{\nu}_{\omega\mu)\lambda}+3M^{\nu}_{(\xi\omega\mu)\lambda}-2D_\xi(R_{\omega\mu(\lambda\alpha)}h^{\nu\alpha})
\end{aligned}
$$

If we contract for ν and ξ and multiply by $h^{\omega\lambda}$ on both sides of the above equation, we find

$$
\begin{aligned}
&-h^{\omega\lambda}D_\beta(R_{\omega\mu\alpha\lambda}h^{\alpha\beta})-h^{\omega\lambda}D_\omega R_{\mu\lambda}+h^{\omega\lambda}D_\mu R_{\omega\lambda}\\
&\quad=-12h^{\omega\lambda}X_{(\beta}H^{\beta}_{\omega\mu)\lambda}+3h^{\omega\lambda}M^{\beta}_{(\beta\omega\mu)\lambda}-2h^{\omega\lambda}D_\beta(R_{\omega\mu(\lambda\alpha)}h^{\alpha\beta})
\end{aligned}
\tag{5.17}
$$

In virtue of (5.1), (5.12), and (5.13), the terms on the left-hand side of (5.17) can be rewritten as

$$
\begin{aligned}
-h^{\omega\lambda}D_\beta(R_{\omega\mu\alpha\lambda}h^{\alpha\beta})&=-D_\beta(R_{\omega\mu\alpha\lambda}h^{\alpha\beta}h^{\omega\lambda})+R_{\omega\mu\alpha\lambda}h^{\alpha\beta}D_\beta(h^{\omega\lambda})\\
&=-D_\beta R^{\beta}_{\mu}+2R_{\omega\mu\alpha\lambda}h^{\alpha\beta}(-X^{\omega}g^{\lambda}_{\beta}+X_\beta h^{\omega\lambda})\\
&=-D_\alpha R^{\alpha}_{\mu}+2X^{\alpha}(R_{\mu\alpha}-V_{\alpha\mu}-R^{\beta}_{\alpha\mu\gamma}k^{\gamma}_{\beta}) \qquad (5.18a)\\
-h^{\omega\lambda}D_\omega R_{\mu\lambda}&=-D_\omega R^{\omega}_{\mu}+R_{\mu\lambda}(D_\omega h^{\omega\lambda})\\
&=-D_\alpha R^{\alpha}_{\mu}+(S^{\alpha}+U^{\alpha})R_{\mu\alpha} \qquad (5.18b)\\
h^{\omega\lambda}D_\mu R_{\omega\lambda}&=D_\mu R-R_{\omega\lambda}D_\mu h^{\omega\lambda}\\
&=D_\mu R+2X^{\alpha}(R_{\alpha\mu}-R_{\alpha\beta}k^{\beta}_{\mu})-2RX_\mu \qquad (5.18c)
\end{aligned}
$$

On the other hand, the relations (5.5a) and (5.11) allows the first term on the right-hand side of (5.17) to be expressed in the form

$$
-12h^{\omega\lambda}X_{(\beta}H^{\beta}_{\omega\mu)\lambda}=8X_\alpha E^{\alpha}_{\mu} \tag{5.18d}
$$

We now substitute (5.18) into (5.17) to complete the proof of (5.16). ■

Remark 5.12. Comparing the expressions (5.15a) and (5.16a), we note that the former is more refined, because the last two terms of (5.16a) are not surveyable.

6. FIELD EQUATIONS IN SEX$_n$

By *field equations* we mean a set of partial differential equations for $g_{\lambda\mu}$. In the present section and in what follows we are concerned with the geometry of field eqations in SEX$_n$ and not with their physical applications.

Chung *et al.* (1987) found the unique SE connection $\Gamma^{\nu}_{\lambda\mu}$ in SEX$_n$ as a function of $g_{\lambda\mu}$ in the form (2.25). Substituting it into (2.10), we saw in the previous two sections that the SE curvature tensor $R^{\nu}_{\omega\mu\lambda}$ together with its contracted curvature tensor $R_{\mu\lambda}$ appear as a function of $g_{\lambda\mu}$. In order to obtain the tensor $g_{\lambda\mu}$ with which we started in dealing with Principles A and B, we prescribe the following conditions for it in terms of $R_{\mu\lambda}$ (see Remark 6.2):

$$R_{(\mu\lambda)} = \partial_{(\mu} Y_{\lambda)} \tag{6.1a}$$

$$R_{(\mu\lambda)} = 0 \tag{6.1b}$$

where Y_λ is an arbitrary vector. Clearly, (6.1a), (6.1b) represent a system of n^2 differential equations of the second order for $g_{\lambda\mu}$.

Therefore, our unified field theory in the n-dimensional SE manifold SEX_n is governed by the following set of equations: n^3 equations (2.7) under the condition (2.24), which determine the unique SE connection $\Gamma^{\nu}_{\lambda\mu}$, and n^2 field equations (6.1) for n^2 unknowns $g_{\lambda\mu}$ (see Theorem 6.5, which states that the unknowns Y_λ are uniquely determined in SEX_n).

Remark 6.1. We note that in the unified field theory of SEX_n the conditions (6.1) are of a purely geometrical nature and physical interpretation is not involved in them *a priori*.

Remark 6.2. Einstein suggested several different sets of field equations in his *four-dimensional unified field theory*. His final suggestion consists of three sets of tensorial differential equations, the first of which is $S_\lambda = 0$. Hlavatý formulated Einstein's idea mathematically by giving 64 equations (2.7) determining the Einstein connection $\Gamma^{\nu}_{\lambda\mu}$ and 20 field equations (2.9) for 20 unknowns $g_{\lambda\mu}$ and X_λ.

Therefore, it would seem natural to follow the analogy of Einstein's field equations (2.9) in our manifold SEX_n, too. However, the restriction $S_\lambda = 0$ is too *strong* in our unified field theory in the SE manifold SEX_n, since this condition implies

$$X_\lambda = 0 \qquad \text{and hence} \quad \Gamma^{\nu}_{\lambda\mu} = \{^{\ \nu}_{\lambda\mu}\}$$

in virtue of (2.25) and (3.2). Therefore, we shall not adopt (2.9) as a starting point, *exclude the condition* $S_\lambda = 0$, and impose the field equations in SEX_n as given in (6.1).

Agreement 6.3. In our further considerations we restrict ourselves to the conditions

$$X_\lambda \neq 0 \qquad \text{and } X_\lambda \text{ not a gradient vector} \tag{6.2}$$

This restriction is quite natural in view of (6.1) and Remark 6.2.

Our first consequence of (6.2) is the following theorem.

Theorem 6.4. In SEX$_n$ we have

$$U^{\nu}_{\lambda\mu} \neq 0 \tag{6.3}$$

Proof. Assume that $U^{\nu}_{\lambda\mu}=0$. Then (3.4) implies that

$$k_{\lambda\nu}X_{\mu}+k_{\mu\nu}X_{\lambda}=0 \qquad \text{for every} \quad \lambda, \mu, \nu$$

In virtue of the condition (6.2), there exists at least one *fixed index* ξ such that $X_{\xi}\neq 0$. Hence

$$k_{\lambda\nu}X_{\xi}+k_{\xi\nu}X_{\lambda}=0 \qquad \text{for every } \lambda \text{ and } \nu \tag{6.4}$$

Putting $\lambda=\xi$ in (6.4), we have $k_{\xi\nu}=0$ for every ν. If $\lambda\neq\xi$, then $k_{\lambda\nu}=0$ for every ν, since $k_{\xi\nu}=0$. Hence we have

$$k_{\lambda\nu}=0 \qquad \text{for every } \lambda \text{ and } \nu$$

which is a contradiction to the nonsymmetry of $g_{\lambda\mu}$. ■

Theorem 6.5. In SEX$_n$, $n\neq 3$, the field equation (6.1a) is satisfied by a unique vector Y_{λ} given by

$$Y_{\lambda}=(3-n)X_{\lambda}=\frac{3-n}{n-1}\,{}^{*}h^{\alpha\beta}\nabla_{\alpha}k_{\beta\lambda} \tag{6.5}$$

Proof. In virtue of (5.8), we have

$$R_{(\mu\lambda)}=(3-n)\partial_{(\mu}X_{\lambda)}$$

from which the first equality follows. The second representation is an immediate consequence of (2.26). ■

Theorem 6.6. In SEX$_n$, $n\neq 3$, the field equation (6.1b) is equivalent to

$$H_{\mu\lambda}+\nabla_{(\mu}T_{\lambda)}-\nabla_{\alpha}U^{\alpha}_{\mu\lambda}+(1-n-2K_2)X_{\mu}X_{\lambda}+U_{\mu}U_{\lambda}-2U_{(\mu}S_{\lambda)}=0 \tag{6.6}$$

Proof. Our assertion (6.6) is an immediate consequence of (5.4) and (6.1b). ■

7. A PARTICULAR SOLUTION OF (2.7) AND (6.1)

In this final section we construct and display *one* particular solution of (2.7) and (6.1) in SEX$_n$ under the condition (2.24).

Agreement 7.1. In our further considerations we restrict ourselves to the cases $n\geq 4$.

Let $h_{\lambda\mu}$ be of the form

$$(h_{\lambda\mu}) = \begin{pmatrix} +1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & +1 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & +1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 \end{pmatrix} \tag{7.1}$$

Let ϕ and θ be two arbitary functions of the argument

$$z = x^{n-1} - x^n \tag{7.2a}$$

possessing at least the first two derivatives with respect to z and satisfying

$$\phi \neq 0, 1 \tag{7.2b}$$

Introduce two vectors

$$A_\lambda = (0, 0, \cdots, 0, 1, -1), \qquad V_\lambda = V_\lambda(z) \tag{7.3}$$

which satisfy the conditions

$$A_\alpha A^\alpha = 0, \qquad A_\alpha V^\alpha = \phi, \qquad V_\alpha V^\alpha = \theta \tag{7.4}$$

Then the tensor field $g_{\lambda\mu} = h_{\lambda\mu} + k_{\lambda\mu}$ with

$$k_{\lambda\mu} = 2A_{(\lambda} V_{\mu)} \tag{7.5}$$

obviously satisfies

$$\mathfrak{h} \neq 0, \qquad \mathfrak{t} = 0, \qquad g \neq 0 \tag{7.6a}$$

and

$$\text{all} \quad \left\{ {}^{\nu}_{\lambda\mu} \right\} = 0 \tag{7.6b}$$

Furthermore, for an arbitrary tensor field $T^{\cdots}_{\cdots}$ that is a function of z, we have in virtue of (7.1), (7.3), and (7.6b)

$$\nabla_\mu T^{\cdots}_{\cdots} = \partial_\mu T^{\cdots}_{\cdots} = A_\mu (T^{\cdots}_{\cdots})' \tag{7.7}$$

where the prime indicates derivative with respect to z.

Agreement 7.2. The situations stated in (7.1)-(7.5) are called "*present conditions*" in our further considerations.

The following sequence of theorems will be proved under present conditions.

Theorem 7.3. Under present conditions we have

$${}^{(p)}k^{\lambda\nu} = \phi^{p-2+\varepsilon} \, {}^{(2-\varepsilon)}k^{\lambda\nu} \qquad (p = 1, 2, 3, \cdots) \tag{7.8a}$$

where

$$^{(2)}k^{\lambda\nu} = 2\phi A^{(\lambda}V^{\nu)} - \theta A^{\lambda}A^{\nu} \tag{7.8b}$$

and

$$\varepsilon = \begin{cases} 0 & \text{if } \quad p \text{ is even} \\ 1 & \text{if } \quad p \text{ is odd} \end{cases} \tag{7.9}$$

Proof. In virtue of (7.4), (7.5), and (2.13), we can derive the relation (7.8b) as follows:

$$\begin{aligned} ^{(2)}k^{\lambda\nu} &= k^{\lambda\alpha}k_{\alpha}^{\nu} = (A^{\lambda}V^{\alpha} - A^{\alpha}V^{\lambda})(A_{\alpha}V^{\nu} - A^{\nu}V_{\alpha}) \\ &= 2\phi A^{(\lambda}V^{\nu)} - \theta A^{\lambda}A^{\nu} \end{aligned}$$

The assertion (7.8a) will be proved by induction on p. In virtue of (7.5) and (7.8b), it can be easily seen that (7.8a) holds for $p = 1, 2$. Now, assume that (7.8a) holds for an arbitrary p. Then, according to the inductive hypothesis, the following relation holds for $q = p + 1$:

$$\begin{aligned} ^{(q)}k^{\lambda\nu} &= {}^{(p)}k^{\lambda\alpha}k_{\alpha}^{\nu} \\ &= \phi^{p-2+\varepsilon}\, {}^{(2-\varepsilon)}k^{\lambda\alpha}k_{\alpha}^{\nu} = \phi^{p-2+\varepsilon}\, {}^{(3-\varepsilon)}k^{\lambda\nu} \\ &= \begin{cases} \phi^{p-2}\, {}^{(3)}k^{\lambda\nu} = \phi^{p}k^{\lambda\nu} = \phi^{q-1}k^{\lambda\nu} & \text{if } p \text{ is even} \\ \phi^{p-1}\, {}^{(2)}k^{\lambda\nu} = \phi^{q-2}\, {}^{(2)}k^{\lambda\nu} & \text{if } p \text{ is odd} \end{cases} \end{aligned}$$

or equivalently

$$^{(q)}k^{\lambda\nu} = \phi^{q-2+\varepsilon'}\, {}^{(2-\varepsilon')}k^{\lambda\nu}; \qquad \varepsilon' = \begin{cases} 0 & \text{if } q \text{ is even} \\ 1 & \text{if } q \text{ is odd} \end{cases}$$

This shows that our assertion (7.8a) holds for $q = p + 1$. ■

Remark 7.4. Note in particular that

$$^{(p)}k^{\lambda\nu} = \phi^{p-2}(2\phi A^{(\lambda}V^{\nu)} - \theta A^{\lambda}A^{\nu}) \qquad \text{when } p \text{ is even} \tag{7.10}$$

Theorem 7.5. Under present conditions we have

$$g\,{}^*h^{\lambda\nu} = \bar{K}_{n-2+\sigma}h^{\lambda\nu} + a\,{}^{(2)}k^{\lambda\nu} \tag{7.11}$$

where

$$a = \sum_{p=0}^{n-4+\sigma} \phi^{n-4+\sigma-p}\bar{K}_p \tag{7.12}$$

Proof. In virtue of (2.15), (2.33a), (7.10), and (7.12), our assertion follows:

$$g\,{}^*h^{\lambda\nu} = \bar{K}_{n-2+\sigma}h^{\lambda\nu} + \bar{K}_{n-4+\sigma}\,{}^{(2)}k^{\lambda\nu} + \bar{K}_{n-6+\sigma}\,{}^{(4)}k^{\lambda\nu} + \cdots + \bar{K}_2\,{}^{(n-4+\sigma)}k^{\lambda\nu}$$
$$+ \bar{K}_0\,{}^{(n-2+\sigma)}k^{\lambda\nu}$$
$$= \bar{K}_{n-2+\sigma}h^{\lambda\nu} + (\bar{K}_{n-4+\sigma} + \bar{K}_{n-6+\sigma}\phi^2 + \cdots + \bar{K}_2\phi^{n-6+\sigma}$$
$$+ \bar{K}_0\phi^{n-4+\sigma})\,{}^{(2)}k^{\lambda\nu}$$
$$= \bar{K}_{n-2+\sigma}h^{\lambda\nu} + a\,{}^{(2)}k^{\lambda\nu} \quad \blacksquare$$

Theorem 7.6. Under present conditions we have

$$X_\lambda = NA_\lambda \tag{7.13}$$

where

$$N = \frac{\phi'(\bar{K}_{n-2+\sigma} + a\phi^2)}{g(1-n)} \tag{7.14}$$

Proof. In virtue of (7.5) and (7.7), it follows from (2.26) that

$$X_\lambda = \frac{1}{n-1}\,{}^*h^{\alpha\beta}\nabla_\alpha k_{\beta\lambda} = \frac{1}{n-1}\,{}^*h^{\alpha\beta}A_\alpha(A_\beta V'_\lambda - A_\lambda V'_\beta)$$

which is equivalent to

$$g(n-1)X_\lambda = (\bar{K}_{n-2+\sigma}h^{\alpha\beta} + a\,{}^{(2)}k^{\alpha\beta})A_\alpha(A_\beta V'_\lambda - A_\lambda V'_\beta)$$
$$= -\phi'(\bar{K}_{n-2+\sigma} + a\phi^2)A_\lambda$$

in virtue of (7.4) and (7.11). Our assertion immediately follows from the above equation. ■

Theorem 7.7. Under present conditions we have

$$A_\alpha X^\alpha = A_\alpha S^\alpha = 0 \tag{7.15a}$$

$$V_\alpha X^\alpha = \phi N \tag{7.15b}$$

$$U_\lambda = \phi N A_\lambda \tag{7.15c}$$

$$\nabla_\mu S_\lambda = (1-n)N' A_\mu A_\lambda, \qquad \nabla_\mu U_\lambda = (\phi N)' A_\mu A_\lambda \tag{7.15d}$$

$$\nabla_\alpha \overset{\alpha}{U}_{\mu\lambda} = 2(\phi N)' A_\mu A_\lambda \tag{7.15e}$$

$$K_2 = -\phi^2 \tag{7.15f}$$

Proof. The assertions in this theorem are direct consequences of (7.4), (7.5), (7.7), and (7.13). In the following we give the proof of (7.15e) making use of $A_\alpha k^\alpha_\mu = \phi A_\mu$ and $A_\alpha (k^\alpha_\mu)' = \phi' A_\mu$:

$$\nabla_\alpha U^\alpha_{\mu\lambda} = \partial_\alpha U^\alpha_{\mu\lambda} = A_\alpha(k^\alpha_\mu X_\lambda + k^\alpha_\lambda X_\mu)' = 2(\phi N)' A_\mu A_\lambda \quad \blacksquare$$

Theorem 7.8. A necessary and sufficient condition for the tensor field $g_{\lambda\mu}$ under present conditions to satisfy the field equation (6.1b) is that ϕ is a solution of

$$(1-n-\phi)N'-\phi'N+[3\phi^3+2(n-1)\phi+1-n]N^2=0 \tag{7.16}$$

Proof. This assertion follows from (6.1b) in virtue of (7.15). ■

Theorem 7.9. If equation (7.16) admits a solution ϕ, the n-dimensional SE connection $\Gamma^{\nu}_{\lambda\mu}$ under present condition that satisfies the field equations (6.1) is given by

$$\Gamma^{\nu}_{\lambda\mu}=2N\delta^{\nu}_{(\lambda}A_{\mu)}+2N(V^{\nu}A_{(\lambda}-A^{\nu}V_{(\lambda})A_{\mu)} \tag{7.17}$$

Proof. This assertion follows from (2.25) in virtue of (7.5), (7.6b), and (7.13). ■

Remark 7.10. Let b and c be arbitrary functions of z with at least the first two derivatives with respect to z. Taking the vector V_λ as

$$V_\lambda=\left(\frac{zb}{(b^2+c^2)^{1/2}},\frac{zc}{(b^2+c^2)^{1/2}},0,0,\cdots,z\right) \tag{7.18a}$$

we have

$$\phi=z,\qquad \phi'=1,\qquad \theta=0 \tag{7.15b}$$

After a lengthy computation, we can prove that $\phi=z$ is a solution of (7.16). Hence, we conclude that *the set of fields $g_{\lambda\mu}$ under present conditions that satisfy the field equations* (6.1) *is not an empty set.*

Remark 7.11. Hlavatý (1954) displayed a particular solution of the Einstein connection $\Gamma^{\nu}_{\lambda\mu}$ that satisfies the field equations (2.9) when $n=4$. Instead of the conditions (7.4) imposed on the vector V_λ, he gave the conditions

$$A_\alpha A^\alpha=A_\alpha V^\alpha=V_\alpha V^\alpha-1=0 \tag{7.19}$$

However, in our SE manifold SEX_4 the conditions (7.19) lead to $X_\lambda=0$, a contradiction to (6.2), and they give no results. Therefore, we need to construct a *new* systems of the tensor field $g_{\lambda\mu}$ under present conditions.

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